

Normed Linear Space: — Let N be a complex (or real) linear space and let $\| \cdot \|$ be a function from N into \mathbb{R} such that for all $x, y \in N$ and all $\alpha \in \mathbb{C}$ (or \mathbb{R}), we have

$$(n_1): \|x\| \geq 0$$

$$(n_2): \|x\| = 0 \Leftrightarrow x = 0$$

$$(n_3): \|x+y\| \leq \|x\| + \|y\|$$

$$(n_4): \|\alpha x\| = |\alpha| \|x\|.$$

Then the function $\| \cdot \|$ is called a norm N and the pair $(N, \| \cdot \|)$ is called a complex (or real) normed linear space.

Theorem: — Let N be a normed linear space and let d be a function from $N \times N$ into \mathbb{R} defined by $d(x, y) = \|x - y\|$. Then d is a metric on N .

Proof: — We shall verify the axioms of above definition.

$$(m_1): x, y \in N \Rightarrow x - y \in N \Rightarrow \|x - y\| \geq 0 \text{ by } (n_1) \\ \Rightarrow d(x, y) \geq 0$$

$$(m_2): d(x, y) = 0 \Rightarrow \|x - y\| = 0 \Rightarrow x - y = 0 \text{ by } (n_2) \\ \Rightarrow x = y.$$

$$(m_3): d(x, y) = \|x - y\| = \|(-1)y - x\| = (-1) \|y - x\| \\ \text{by } (n_4) = \|y - x\| = d(y, x)$$

$$(m_4): \text{if } x, y, z \in N, \text{ then} \\ \|x - y\| = \|x - z + z - y\| \leq \|x - z\| + \|z - y\| \text{ (by } n_3) \\ \text{Hence } d(x, y) \leq d(x, z) + d(z, y)$$

It follows that d is a metric on N .

Banach Space or B-Space:

A normed linear space which is complete as a metric space is called a Banach Space or B-Space.

Example of Banach Spaces

1. Show that the real linear spaces \mathbb{R} and the complex linear space \mathbb{C} are Banach spaces under the norm

$$\|x\| = |x|; x \in \mathbb{C} \text{ (or } \mathbb{R}\text{)}$$

Solution: — Since $\|x\| = |x| \geq 0$ and $\|x\| = 0 \Rightarrow |x| = 0 \Rightarrow x = 0$ (Axioms (n1), and (n2))

(n3): Let $z, w \in \mathbb{C}$ and let \bar{z} and \bar{w} denote their complex conjugates, then

$$|z+w|^2 = (z+w)(\bar{z}+\bar{w})$$

$$= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \leq |z|^2 + 2|z||w| + |w|^2$$

$$|w|^2 = |z|^2 + 2|z||w| + |w|^2$$

$$(\because |z\bar{w}| = |z\bar{w}| = |z||w|)$$

$$= (|z| + |w|)^2$$

Hence $|z+w| \leq |z| + |w|$ that is $\|z+w\| \leq$

$$\|z\| + \|w\|.$$

(n4): $\|\alpha x\| = |\alpha x| = |\alpha||x| = |\alpha|\|x\|.$

Thus \mathbb{C} and \mathbb{R} are both normed linear spaces. Also we know that these spaces are complete. Hence they are Banach spaces.

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